

A NOTE ON TRIANGLE PARTITIONS

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Abstract. Koivisto studied the partitioning of sets of bounded cardinality. We improve his time analysis somewhat, for the special case of triangle partitions, and obtain a slight improvement.

1 Introduction

Let $G = (V, E)$ be a graph. The triangle partition problem asks to partition the vertices of G into vertex-disjoint triangles.

The triangle partition problem is NP-complete, even for graphs with maximal degree at most four [3]. Via the inclusion-exclusion method the problem can be solved in $O^*(2^n)$ time and polynomial space [1].

Koivisto analyzes the time complexity of partitioning a set into subsets of bounded cardinality. His method shows that the triangle partition problem can be solved in $O^*(1.7693^n)$.

In this note we simplify his analysis somewhat, for the case of triangle partitions, and obtain an $O^*(1.7549^n)$ algorithm.

2 Triangle partitions

Let $G = (V, E)$ be a graph. Let $V = \{1, \dots, n\}$. When G has a triangle partition then $q = \frac{n}{3}$ is integer.

Koivisto's idea is to search for triangles that are in lexicographic order.

Lemma 1. *Consider the lexicograph ordering of the triangles in a triangle partition \mathcal{P} of G . Then, for $j \in \{1, \dots, q\}$, the first j triangles in \mathcal{P} must contain the vertices of $\{1, \dots, j\}$.*

Proof. Since the triangles of \mathcal{P} partition V each element of V is in some triangle of \mathcal{P} . Since the triangles are lexicographically ordered, the first j triangles of \mathcal{P} must contain all vertices of $\{1, \dots, j\}$. \square

It follows that the first j triangles must contain $\{1, \dots, j\}$ and the remaining vertices of these triangles are $2j$ vertices from $\{j+1, \dots, n\}$. Koivisto shows that a dynamic programming algorithm can be obtained that runs in time proportional to

$$\sum_{k=1}^{n/3} \binom{n-k}{2k}.$$

Via the Hoeffding bound Koivisto obtains his (general) result. In the following lemma we simplify the analysis and obtain a slightly better bound for the case of triangle partitions.

Lemma 2.

$$\sum_{k=1}^{n/3} \binom{n-k}{2k} = O^*(1.7549^n).$$

Proof. Let $q = \frac{n}{3}$.

The binomial coefficients can be bounded as follows. Write

$$k = \alpha n \quad \text{and} \quad \beta = \frac{2\alpha}{1-\alpha}. \quad (1)$$

First consider the tails. When $\beta \leq \frac{1}{3}$ then $\alpha \leq \frac{1}{7}$ and we have

$$\binom{n-k}{2k} \leq 2^{2n/7} \leq 1.22^n.$$

When $\beta \geq \frac{2}{3}$ then $\alpha \geq \frac{1}{4}$ and we find

$$\binom{n-k}{2k} = \binom{n-k}{n-3k} \leq \binom{n}{n-3k} \leq 2^{n/4} \leq 1.2^n.$$

Consider the case where $\frac{1}{3} < \beta < \frac{2}{3}$, that is, $\frac{1}{7} < \alpha < \frac{1}{4}$. Write $\beta = \frac{1+\epsilon}{2}$. Then $-\frac{1}{3} < \epsilon < \frac{1}{3}$. With Stirling's formula we obtain (neglecting polynomial factors)

$$\binom{n-k}{2k} \sim \exp \left[\frac{2n}{5+\epsilon} (2 \ln(2) - (1-\epsilon) \ln(1-\epsilon) - (1+\epsilon) \ln(1+\epsilon)) \right]. \quad (2)$$

Now write $\gamma = \frac{1-\epsilon}{2}$. Then $\frac{1}{3} < \gamma < \frac{2}{3}$ and (2) becomes

$$\binom{n-k}{2k} \sim \exp \left[\frac{2n}{3-\gamma} f(\gamma) \right] \quad \text{where} \quad \gamma = \frac{1-3\alpha}{1-\alpha} \quad \text{and} \quad \alpha = \frac{k}{n} \quad (3)$$

and where $f(\gamma) = -\gamma \ln(\gamma) - (1-\gamma) \ln(1-\gamma)$ is the entropy function.

Define

$$g(\gamma) = \frac{1}{3-\gamma} \cdot f(\gamma). \quad (4)$$

The function $g(\gamma)$ has its maximum at $\gamma \leq 0.56985$ and with this value (3) becomes

$$\binom{n-k}{2k} = O^*(1.7549^n). \quad (5)$$

□

Theorem 1. *There exists an $O^*(1.7549^n)$ time algorithm that check if the vertices of a graph can be partitioned into vertex-disjoint triangles.*

Proof. This follows from Lemma 2 and Koivisto's dynamic programming algorithm. □

References

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